# Dissipation enhancement for stochastic heat equation with transport noise

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Based on arXiv:2104.01740

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16th Workshop on Markov Processes and Related Topics

BNU & CSU, 2021.7.12

# Outline

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1. Backgrounds
2. Main results
3. Sketch of proofs

# Stabilization by noise

- For a $d \times d$ matrix $A_0$, the trivial solution $X_t \equiv 0$ of ODE

$$\dot{X}_t = A_0 X_t, \quad X_0 \in \mathbb{R}^d$$

  is unstable if $A_0$ has positive eigenvalues.

- When can the above system be stabilized by noise?

$$\mathrm{d}X_t = A_0 X_t \,\mathrm{d}t + \sum_{i=1}^{m} A_i X_t \circ \mathrm{d}W_t^i, \quad X_0 \in \mathbb{R}^d.$$

  Here $A_1, \ldots, A_m$ are some matrices.

- Define the Lyapunov exponent

$$\lambda(X_0, \omega) = \limsup_{t\to\infty} \frac{1}{t} \log |X_t(X_0, \omega)|.$$

  Oseledec's *multiplicative ergodic theorem*: with prob. 1, $\exists\, d$ random numbers $\lambda_1 \leq \cdots \leq \lambda_d$ such that $\lambda(X_0, \omega)$ takes one of them.

# Stabilization by noise

- L. Arnold-Crauel-Wihstutz (SIAM J Contr. Optim., 1983): for some $\nu > 0$, consider

$$\mathrm{d}X_t = A_0 X_t \,\mathrm{d}t + \nu \sum_{i=1}^{m} A_i X_t \circ \mathrm{d}W_t^i, \quad X_0 \in \mathbb{R}^d.$$

$\nu$ is the intensity of the noise.

**Theorem**

*There are skew-symmetric matrices $A_1, \ldots, A_m$ such that the top Lyapunov exponent $\lambda_\nu$ satisfies*

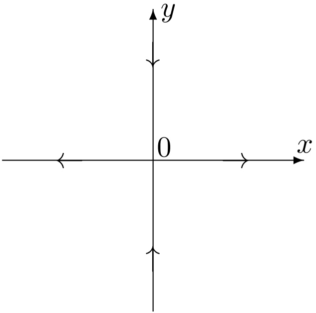
$$\lim_{\nu \to \infty} \lambda_\nu = \frac{\mathrm{Tr}(A_0)}{d}.$$

*In particular, $\dot{X}_t = A_0 X_t$ can be stabilized by noise iff $\mathrm{Tr}(A_0) < 0$.*

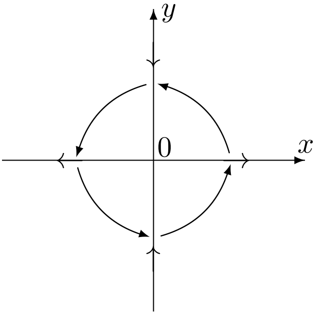
$$\mathrm{d}|X_t|^2 = 2\langle X_t, A_0 X_t\rangle \,\mathrm{d}t + 2\nu \sum_{i=1}^{m} \underbrace{\langle X_t, A_i X_t\rangle}_{=0} \circ \mathrm{d}W_t^i = 2\langle X_t, A_0 X_t\rangle \,\mathrm{d}t.$$

# A trivial example

Consider $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, $\mathrm{Tr}(A_0) = -1 < 0.$




$\dot{X}_t = A_0 X_t$




$\mathrm{d}X_t = A_0 X_t \,\mathrm{d}t + \nu \sum_{i=1}^m A_i X_t \circ \mathrm{d}W_t^i$

Intuition: the noise mixes the two coordinates to yield stability on average.

# Transport noise for parabolic equations

- M. Capiński's conjecture (around 1987):

  In infinite dimensional case, stochastic transport noise could have a similar stabilizing effect for parabolic equations.

- Heat equation with transport noise:

$$\mathrm{d} f = \Delta f \,\mathrm{d}t + \nu \sum_k \sigma_k \cdot \nabla f \circ \mathrm{d}W_t^k,$$

  where $\{\sigma_k\}_k$ are divergence free vector fields. Since

$$\langle \sigma_k \cdot \nabla f, g \rangle_{L^2} = -\langle f, \sigma_k \cdot \nabla g \rangle_{L^2},$$

  transport noise behaves similarly as skew-symmetric matrices.

- By Stratonovich calculus,

$$\mathrm{d}\|f\|_{L^2}^2 = 2\langle f, \Delta f \rangle \,\mathrm{d}t + 2\nu \sum_k \langle f, \sigma_k \cdot \nabla f \rangle \circ \mathrm{d}W_t^k = -2\|\nabla f\|_{L^2}^2 \,\mathrm{d}t.$$

# Heuristic explanation


- Stirring the fluid produces small scale motions, which correspond to high frequencies in Fourier series.
- Higher frequencies correspond to smaller eigenvalues of $\Delta$ $(-\lambda_k \searrow -\infty)$, which yield stronger dissipation.
- Since transport noise mixes Fourier modes, we expect the top Lyapunov exponent

$$\lim_{\nu \to \infty} \lambda_\nu = -\infty.$$

# Some results in deterministic setting

- Constantin-Kiselev-Ryzhik-Zlatoš (Ann. Math., 2008) considered heat equation on compact manifolds $M$:

$$\partial_t f^\nu = \Delta f^\nu + \nu\, b \cdot \nabla f^\nu, \quad f^\nu(0,x) = f_0(x),$$

where $b$ is a Lipschitz divergence free vector field.

They studied the phenomenon of dissipation enhancement.

**Theorem**

*If $b \cdot \nabla$ has no eigenfunctions in $H^1(M)$ (except trivial constant functions), then $\forall\, t > 0$ and $f_0 \in L^2(M)$, one has*

$$\lim_{\nu \to \infty} \big\| f^\nu(t,\cdot) - \bar f_0 \big\|_{L^2} = 0,$$

*where $\bar f_0 = \frac{1}{|M|} \int_M f_0 \, \mathrm{d}x$.*

# Some results in deterministic setting

- A. Zlatoš, Diffusion in fluid flow: dissipation enhancement by flows in 2D. *Commun. Partial Differ. Equ.* 35(3), 496–534 (2010)
- J. Bedrossian, M. Coti Zelati, Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows. *Arch. Rat. Mech. Anal.* 224(3), 1161–1204 (2017)
- Y. Feng, G. Iyer, Dissipation enhancement by mixing. *Nonlinearity* 32(5), 1810–1851 (2019)
- M. Coti Zelati, M.G. Delgadino, T.M. Elgindi, On the relation between enhanced dissipation timescales and mixing rates. *Commun. Pure Appl. Math.* 73(6), 1205–1244 (2020)
- J. Bedrossian, A. Blumenthal, S. Punshon-Smith, Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection -diffusion by stochastic Navier-Stokes. *Probab. Theory Related Fields* 179(3-4), 777–834 (2021)
- B. Gess, I. Yaroslavtsev, Stabilization by transport noise and enhanced dissipation in the Kraichnan model. arXiv:2104.03949 (2021)

# Transport noise in inviscid models

- 2D Euler equation:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = 0 \\ \nabla\cdot u = 0 \end{cases} \longleftrightarrow \begin{cases} \partial_t \xi + u\cdot\nabla\xi = 0 \\ \xi = \nabla^\perp\cdot u = \partial_2 u_1 - \partial_1 u_2 \end{cases}$$

The enstrophy meas. (or white noise meas.) is formally invariant:

$$\mu(\mathrm{d}\xi) = \frac{1}{Z}\exp\Big(-\frac{1}{2}\|\xi\|_{L^2}^2\Big)\,\mathrm{d}\xi, \quad \mathrm{supp}(\mu) = H^{-1-}(\mathbb{T}^2).$$

- Flandoli-Luo (AoP, 2020): white noise solutions of stoch. 2D Euler eqs

$$\mathrm{d}\xi^N + u^N\cdot\nabla\xi^N\,\mathrm{d}t = \sqrt{\frac{2\nu}{\log N}}\sum_{k\in\mathbb{Z}_0^2,|k|\le N}\frac{1}{|k|}\sigma_k\cdot\nabla\xi^N\circ\mathrm{d}W_t^k$$

$$= \nu\Delta\xi^N\,\mathrm{d}t + \sqrt{\frac{2\nu}{\log N}}\sum_{k\in\mathbb{Z}_0^2,|k|\le N}\frac{1}{|k|}\sigma_k\cdot\nabla\xi^N\,\mathrm{d}W_t^k$$

converge to the unique stationary solu. of 2D NSEs with space-time white noise:

$$\partial_t\xi + u\cdot\nabla\xi = \nu\Delta\xi + \sqrt{2\nu}\nabla^\perp\cdot\eta \longleftrightarrow \partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \sqrt{2\nu}\eta,$$

where $\eta(t,x) = \sum_k \sigma_k(x)\dot W_t^k$, $\sigma_k(x) = \frac{k^\perp}{|k|}e_k(x)$.

# Stoch. linear transport eqs

- Lucio Galeati (*SPDE Anal Comp*, 2020): stochastic linear transport eqs with $L^2$-initial data $f_0$:

$$\mathrm{d}f = b\cdot\nabla f\,\mathrm{d}t + \sqrt{2\nu}\sum_{k\in\mathbb{Z}_0^2}\theta_k\sigma_k\cdot\nabla f\circ \mathrm{d}W_t^k$$

$$= b\cdot\nabla f\,\mathrm{d}t + \nu\Delta f\,\mathrm{d}t + \sqrt{2\nu}\sum_{k}\theta_k\sigma_k\cdot\nabla f\,\mathrm{d}W_t^k,$$

where $\nu>0$ is noise intensity, $\{\theta_k\}_k\in\ell^2(\mathbb{Z}_0^2)$, $\theta_k=\theta_l$ for all $|k|=|l|$. $\forall\, t\le T$, it holds $\|f_t\|_{L^2}\le C_T\|f_0\|_{L^2}$ (independent of $\nu$ and $\theta\in\ell^2$).

- Taking $\{\theta^N\}_{N\ge1}\subset\ell^2(\mathbb{Z}_0^2)$ satisfying $\|\theta^N\|_{\ell^2}=1$ and $\lim_{N\to\infty}\|\theta^N\|_{\ell^\infty}=0$, Lucio Galeati proved

$$\mathrm{d}f^N = b\cdot\nabla f^N\,\mathrm{d}t + \nu\Delta f^N\,\mathrm{d}t + \sqrt{2\nu}\sum_{k}\theta_k^N\sigma_k\cdot\nabla f^N\,\mathrm{d}W_t^k$$

converge to the deterministic parabolic eq:

$$\partial_t f = b\cdot\nabla f + \nu\Delta f,\quad f|_{t=0}=f_0.$$

- If $\theta_k^N=\frac{1}{Z_N}\frac{\mathbf{1}_{\{N\le|k|\le 2N\}}}{|k|^a}$ $(k\in\mathbb{Z}_0^2)$, then $Z_N\sim\frac{1}{N^{a-1}}$ and $\|\theta^N\|_{\ell^\infty}\sim\frac{1}{N}$.

# Why the martingale part vanishes?

Recall the martingale part

$$\mathrm{d}M_t^N = \sum_{k\in\mathbb{Z}_0^2} \theta_k^N \sigma_k \cdot \nabla f_t^N \,\mathrm{d}W_t^k.$$

For any $\phi \in C^1(\mathbb{T}^2)$, we have

$$\begin{aligned}
\langle M_t^N, \phi\rangle &= -\sum_k \theta_k^N \int_0^t \langle f_s^N, \sigma_k\cdot\nabla\phi\rangle \,\mathrm{d}W_s^k. \\
\Longrightarrow\ \mathbb{E}\langle M_t^N, \phi\rangle^2 &= \sum_k \big(\theta_k^N\big)^2\, \mathbb{E}\int_0^t \langle f_s^N, \sigma_k\cdot\nabla\phi\rangle^2 \,\mathrm{d}s \\
&\le \|\theta^N\|_{\ell^\infty}^2\, \mathbb{E}\int_0^t \sum_k \langle f_s^N\nabla\phi, \sigma_k\rangle^2 \,\mathrm{d}s \\
&\le \|\theta^N\|_{\ell^\infty}^2\, \mathbb{E}\int_0^t \|f_s^N\nabla\phi\|_{L^2}^2 \,\mathrm{d}s \\
&\le \|\theta^N\|_{\ell^\infty}^2 \|\nabla\phi\|_{L^\infty}^2 C_T \|f_0\|_{L^2}^2 \to 0.
\end{aligned}$$

# Scaling limit of stoch 2D Euler eq

- Flandoli-Galeati-Luo (*J Evol Equ*, 2021):
  stochastic 2D Euler equations in vorticity form, with $L^2$-initial data:

$$\begin{aligned} \mathrm{d}\xi + u\cdot\nabla\xi\,\mathrm{d}t &= \sqrt{2\nu}\sum_k \theta_k\sigma_k\cdot\nabla\xi\circ \mathrm{d}W_t^k \\ &= \nu\Delta\xi\,\mathrm{d}t + \sqrt{2\nu}\sum_k \theta_k\sigma_k\cdot\nabla\xi\,\mathrm{d}W_t^k. \end{aligned}$$

  $\exists$ weak solution $\{\xi_t\}_{t\in[0,T]}$ such that $\mathbb{P}$-a.s. $\|\xi_t\|_{L^2}\le\|\xi_0\|_{L^2}$.

- Taking $\{\theta^N\}_N\subset\ell^2(\mathbb{Z}_0^2)$ s.t. $\|\theta^N\|_{\ell^2}=1$ and $\lim_{N\to\infty}\|\theta^N\|_{\ell^\infty}=0$,
  we proved $\xi^N \xrightarrow{w} \xi$, the unique solution of the deterministic 2D NSE in vorticity form:

$$\begin{cases} \partial_t\xi + u\cdot\nabla\xi = \nu\Delta\xi, \\ u = K*\xi \end{cases} \quad\longleftrightarrow\quad \begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u, \\ \nabla\cdot u = 0. \end{cases}$$

- The proof is based on compactness arguments, hence there is no convergence rate.

# Related works: suppression of blow-up

- 3D NSEs on $\mathbb{T}^3$.

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \Delta u \\ \nabla\cdot u = 0 \end{cases} \quad \longleftrightarrow \quad \begin{cases} \partial_t \xi + u\cdot\nabla \xi - \xi\cdot\nabla u = \Delta \xi \\ \xi = \nabla\times u \end{cases}$$

$u,\, \xi : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}^3$.

- Flandoli-Luo (PTRF, 2021): Stoch 3D NSEs.

$$\mathrm{d}\xi + (u\cdot\nabla\xi - \xi\cdot\nabla u)\,\mathrm{d}t = \Delta\xi\,\mathrm{d}t + \sqrt{2\nu}\sum_{k\in\mathbb{Z}_0^3}\sum_{i=1}^{2}\theta_k \Pi(\sigma_{k,i}\cdot\nabla\xi)\circ \mathrm{d}W_t^{k,i}.$$

$\forall\, R>0$, $\exists$ big $\nu$ and $\theta\in\ell^2(\mathbb{Z}_0^3)$, such that $\forall\, \|\xi_0\|_{L^2}\le R$, global solution exists with large probability.

- Flandoli-Galeati-Luo (Comm. PDE, 2021): general nonlinear eqs on $\mathbb{T}^d$.

$$\mathrm{d}f = -(-\Delta)^\alpha f\,\mathrm{d}t + \Phi(f)\,\mathrm{d}t + \sqrt{2\nu}\sum_{k\in\mathbb{Z}_0^d}\sum_{i=1}^{d-1}\theta_k\sigma_{k,i}\cdot\nabla f\circ \mathrm{d}W_t^{k,i},$$

where $\alpha\ge 1$ and $\Phi : H^\alpha\to H^{-\alpha}$ is some nonlinear mapping.

# Outline

Heat equation with transport noise on $\mathbb{T}^2$:

$$\partial_t f = \kappa \Delta f + \dot{W}_t \cdot \nabla f, \quad W_t(x) = \sqrt{2\nu} \sum_{k \in \mathbb{Z}_0^2} \theta_k \sigma_k(x) W_t^k;$$

- $\kappa > 0$: molecular diffusivity; $\nu > 0$: noise intensity.
- $\theta = (\theta_k)_{k \in \mathbb{Z}_0^2} \in \ell^2(\mathbb{Z}_0^2)$, $\|\theta\|_{\ell^2} = 1$, $\theta$ is radially symmetric.
- $\sigma_k(x) = \frac{k^\perp}{|k|} e_k(x)$, $k \in \mathbb{Z}_0^2$.

Stratonovich to Itô:

$$\begin{aligned} \mathrm{d}f &= \kappa \Delta f \,\mathrm{d}t + \sqrt{2\nu} \sum_k \theta_k \sigma_k \cdot \nabla f \circ \mathrm{d}W_t^k \\ &= \kappa \Delta f \,\mathrm{d}t + \nu \sum_k \theta_k^2 \sigma_k \cdot \nabla(\sigma_k \cdot \nabla f) \,\mathrm{d}t + \sqrt{2\nu} \sum_k \theta_k \sigma_k \cdot \nabla f \,\mathrm{d}W_t^k \\ &= (\kappa + \nu) \Delta f \,\mathrm{d}t + \sqrt{2\nu} \sum_k \theta_k \sigma_k \cdot \nabla f \,\mathrm{d}W_t^k. \end{aligned}$$

# Dissipation enhancement

- Though there is $\nu\Delta$, it does not mean dissipation has already been enhanced.
- Indeed, by Itô's formula,

$$\begin{aligned} \mathrm{d}\|f\|_{L^2}^2 &= 2\kappa\langle f, \Delta f\rangle\,\mathrm{d}t + 2\sqrt{2\nu}\sum_k \theta_k \langle f, \sigma_k\cdot\nabla f\rangle \circ \mathrm{d}W_t^k \\ &= -2\kappa\|\nabla f\|_{L^2}^2\,\mathrm{d}t \le -8\kappa\pi^2\|f\|_{L^2}^2\,\mathrm{d}t \\ \Longrightarrow \|f_t\|_{L^2} &\le \|f_0\|_{L^2}\, e^{-4\kappa\pi^2 t} \quad \text{(independent of } \nu \text{ and } \theta) \end{aligned}$$

**Theorem (Flandoli-Galeati-Luo, arXiv:2104.01740)**

*$\forall\, \lambda > 0$, $\exists\, \nu > 0$ and $\theta \in \ell^2(\mathbb{Z}_0^2)$ with the following property:*

*$\forall\, f_0 \in L^2(\mathbb{T}^d)$ with zero mean, $\exists$ a random constant $C > 0$ such that, for the solution $f_t$ with initial condition $f_0$, it holds*

$$\mathbb{P}\text{-}a.s. \quad \|f_t\|_{L^2} \le C e^{-\lambda t} \quad \textit{for all } t \ge 0.$$

# Outline

1. Backgrounds
2. Main results
3. Sketch of proofs

# Stoch heat eqn and mild formulation

$$\begin{aligned} \mathrm{d}f &= \kappa\Delta f\,\mathrm{d}t + \sqrt{2\nu}\sum_k \theta_k\sigma_k\cdot\nabla f\circ \mathrm{d}W_t^k \\ &= (\kappa+\nu)\Delta f\,\mathrm{d}t + \sqrt{2\nu}\sum_k \theta_k\sigma_k\cdot\nabla f\,\mathrm{d}W_t^k. \end{aligned}$$

- Energy balance: $\forall\, 0\le s<t$,

$$\|f_t\|_{L^2}^2 + 2\kappa\int_s^t \|\nabla f_r\|_{L^2}^2\,\mathrm{d}r = \|f_s\|_{L^2}^2,$$

  in particular, $t\to\|f_t\|_{L^2}^2$ is decreasing.

- Mild formulation: $\forall\, 0\le s<t$,

$$f_t = e^{(\kappa+\nu)(t-s)\Delta}f_s + \underbrace{\sqrt{2\nu}\sum_k \theta_k\int_s^t e^{(\kappa+\nu)(t-r)\Delta}(\sigma_k\cdot\nabla f_r)\,\mathrm{d}W_r^k}_{Z_{s,t}}.$$

# Key lemma

## Lemma

*$\exists\, \delta \in (0,1)$ such that for any $n \geq 0$,*

$$\mathbb{E}\|f_{n+1}\|_{L^2}^2 \leq \delta\, \mathbb{E}\|f_n\|_{L^2}^2,$$

*where, for some $0 < \alpha < 1 < \beta$,*

$$\delta \leq C_{\alpha,\beta}\Big[(\kappa+\nu)^{-1} + \nu^{\frac{4\beta-\alpha(2\beta+d)}{4(\alpha+\beta)}}\kappa^{-\frac{\beta}{\alpha+\beta}}\|\theta\|_{\ell^\infty}^{\frac{2\alpha}{\alpha+\beta}}\Big].$$

*In particular, $\delta$ can be as small as possible by first taking $\nu$ big, and then letting $\|\theta\|_{\ell^\infty}$ small enough.*

Idea of proof: using monotonicity $t \to \|f_t\|_{L^2}^2$ and mild formulation $f_t = e^{(\kappa+\nu)(t-s)\Delta} f_s + Z_{s,t}$,

$$\begin{aligned}\|f_{n+1}\|_{L^2}^2 \leq \int_n^{n+1}\|f_t\|_{L^2}^2\,\mathrm{d}t &\leq 2\int_n^{n+1}\|e^{(\kappa+\nu)(t-n)\Delta} f_n\|_{L^2}^2\,\mathrm{d}t + \int_n^{n+1}\|Z_{n,t}\|_{L^2}^2\,\mathrm{d}t \\ &\leq 2\int_n^{n+1} e^{8\pi^2(\kappa+\nu)(t-n)}\|f_n\|_{L^2}^2\,\mathrm{d}t + \int_n^{n+1}\|Z_{n,t}\|_{L^2}^2\,\mathrm{d}t.\end{aligned}$$

# Proof of the main theorem

**Theorem (Flandoli-Galeati-Luo, arXiv:2104.01740)**

$\forall\, \lambda > 0$, $\exists\, \nu > 0$ *and* $\theta \in \ell^2(\mathbb{Z}_0^2)$ *with the following property:*

$\forall\, f_0 \in L^2(\mathbb{T}^d)$ *with zero mean,* $\exists$ *a random constant* $C > 0$ *such that, for the solution* $f_t$ *with initial condition* $f_0$*, it holds*

$$\mathbb{P}\text{-}a.s. \quad \|f_t\|_{L^2} \le C e^{-\lambda t} \quad \textit{for all } t \ge 0.$$

**Proof**. The above lemma implies $\exists\, \delta \in (0,1)$ such that for any $n \ge 1$,

$$\begin{aligned} \mathbb{E}\bigg[\sup_{t\in[n,n+1]} \|f_t\|_{L^2}^2\bigg] = \mathbb{E}\|f_n\|_{L^2}^2 &\le \delta\, \mathbb{E}\|f_{n-1}\|_{L^2}^2 \le \cdots \\ &\le \delta^n \|f_0\|_{L^2}^2 = e^{-\lambda_0 n}\|f_0\|_{L^2}^2. \end{aligned}$$

Choosing $\nu$ big and $\|\theta\|_{\ell^\infty}$ small, we can make $\lambda_0$ as big as we want.

# Proof of the main theorem

Fix any $\lambda \in (0, \lambda_0)$, define the event

$$A_n = \Big\{\omega \in \Omega : \sup_{t\in[n,n+1]} \|f_t(\omega)\|_{L^2}^2 > e^{-\lambda n}\Big\}, \quad n \geq 1.$$

Then by Chebyshev's inequality,

$$\sum_n \mathbb{P}(A_n) \leq \sum_n e^{\lambda n} \mathbb{E}\Big[\sup_{t\in[n,n+1]} \|f_t\|_{L^2}^2\Big] \leq \sum_n e^{(\lambda-\lambda_0)n} \|f_0\|_{L^2}^2 < +\infty.$$

Borel-Cantelli lemma implies, for $\mathbb{P}$-a.s. $\omega$, $\exists\, n(\omega) \in \mathbb{N}$ such that

$$\sup_{t\in[n,n+1]} \|f_t(\omega)\|_{L^2}^2 \leq e^{-\lambda n}, \quad \forall\, n \geq n(\omega).$$

Thus $\exists$ a random constant $C > 0$ such that

$$\|f_t(\omega)\|_{L^2}^2 \leq C e^{-\lambda t}, \quad t \geq 0.$$

谢谢大家!